

Intersectionless Envelope Estimation for EMD

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- [HK13] Boqiang Huang and Angela Kunoth. 'An optimization based empirical mode decomposition scheme'. In: *Journal of Computational and Applied Mathematics* 240 (Mar. 2013), pp. 174–183. DOI: 10.1016/j.cam.2012.07.012.
- [HPH12] Xiyuan Hu, Silong Peng and Wen-Liang Hwang. 'EMD Revisited: A New Understanding of the Envelope and Resolving the Mode-Mixing Problem in AM-FM Signals'. In: *IEEE Transactions on Signal Processing* 60.3 (Mar. 2012), pp. 1075–1086. DOI: 10.1109/TSP.2011.2179650.
- [Hua+98] Norden E. Huang, Zheng Shen, Steven R. Long, Manli C. Wu, Hsing H. Shih, Quanan Zheng, Yen Nai-Chyuan, Chi Chao Tung and Henry H. Liu. 'The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis'. In: *Proceedings of the Royal Society A*. Vol. 454. 1971. Mar. 1998, pp. 903–995. DOI: 10.1098/rspa.1998.0193.
- [Hun] Laslo Hunhold. In preparation. PhD thesis. University of Cologne.
- [Niu+21] Xiao-dong Niu, Jian Wang, Xing-cheng Han, Xuan Li and Li-ming Wang. 'An Improved Empirical Mode Decomposition Based on Local Integral Mean and Its Application in Signal Processing'. In: *Mathematical Problems in Engineering* 2021 (Feb. 2021). DOI: 10.1155/2021/8891217.

Introduction

Common Methods

Goal: ‘Additive’ decomposition of the multi-component signal $s(t)$

Motivation: Data analysis and -compression

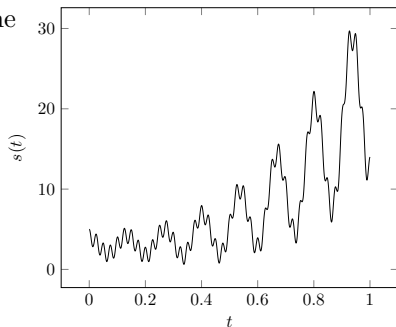
▶ FOURIER-Transformation

- ▶ Decomposition into weighted ω -oscillation-terms $\exp(i\omega t)$
- ▶ ‘Global’ frequency
- ▶ Application: (piecewise) stationary signals

▶ Wavelet-Transformation

- ▶ Decomposition into weighted wavelet basis functions (compact support)
- ▶ ‘Local’ frequency
- ▶ Application: non-linear, non-stationary signals

Problem: Instantaneous frequency not determinable, thus time-frequency resolution fuzzy.



Empirical Mode Decomposition

Intrinsic Mode Function (IMF) [Hua+98] [Hun]

Intrinsic Mode Function (IMF)

$$u(t) := a(t) \cdot \cos(\phi(t))$$

with

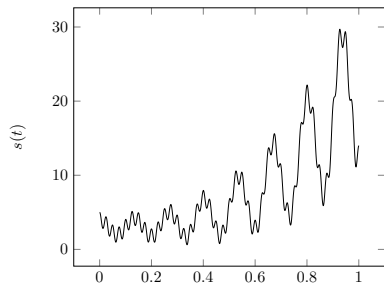
- ▶ $a \geq 0$
- ▶ $\phi' > 0$
- ▶ a, ϕ ‘slowly-varying’

Physical interpretation

- ▶ a : amplitude
- ▶ ϕ : phase
- ▶ ϕ' : frequency

Empirical Mode Decomposition

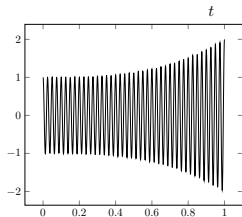
Additive decomposition into Intrinsic Mode Functions (IMFs) [Hua+98]



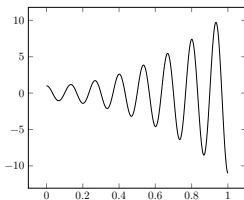
$$s(t) = \left[\sum_{k=0}^M u_k(t) \right] + r_{M+1}(t)$$

$u_k(t)$: Intrinsic Mode Function

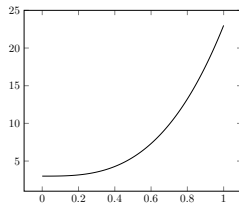
$r_{M+1}(t)$: Residual



(a) $u_0(t)$



(b) $u_1(t)$

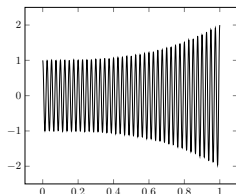


(c) $r_2(t)$

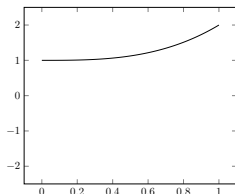
[Hun]

Empirical Mode Decomposition

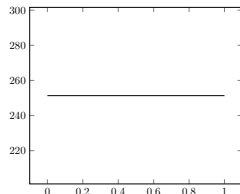
Amplitude and Frequency from IMFs [Hua+98]



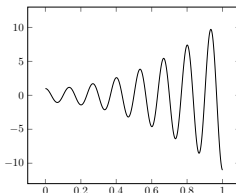
(d) $u_0(t) = a_0(t) \cdot \cos(\phi_0(t))$



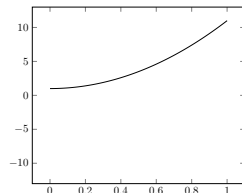
(e) $a_0(t)$



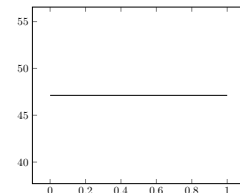
(f) $\phi'_0(t)$



(g) $u_1(t) = a_1(t) \cdot \cos(\phi_1(t))$



(h) $a_1(t)$



(i) $\phi'_1(t)$

Result: Instantaneous frequency and amplitude, sharp time-frequency resolution. [Hun]

Empirical Mode Decomposition

Overview



Subproblem 1 (Additive decomposition into IMFs)

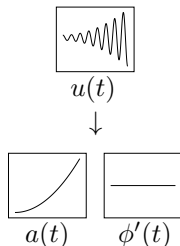
- ▶ ‘Sifting’ [Hua+98][Che+06] or
- ▶ Operator-Based Signal Separation (OSS) [Guo+17]

$$\text{Solve } \min_{u \text{ IMF}} \|s - u\|_2^2$$

Subproblem 2 (Amplitude and frequency from IMFs)

- ▶ Classical method [Hua+98] or
- ▶ Null-Space-Pursuit (NSP) [PH10]:
Use differential operator $\mathcal{D}_{a,\phi}$
with $\mathcal{D}_{a,\phi}(a \cdot \cos(\phi)) = 0$ and solve

$$\min_{a,\phi} \|\mathcal{D}_{a,\phi} u\|_2^2$$



[Guo+17]

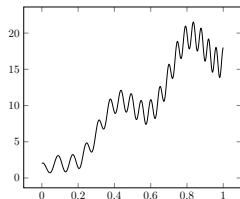
Baokui Guo, Silong Peng, Xiyuan Hu and Pengcheng Xu. ‘Complex-valued differential operator-based method for multi-component signal separation’. In: *Signal Processing* 132 (Mar. 2017), pp. 66–76. DOI: 10.1016/j.sigpro.2016.09.015.

[PH10]

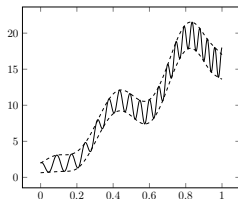
Silong Peng and Wen-Liang Hwang. ‘Null Space Pursuit: An Operator-based Approach to Adaptive Signal Separation’. In: *IEEE Transactions on Signal Processing* 58.5 (Jan. 2010), pp. 2475–2483. DOI: 10.1109/TSP.2010.2041606.

Sifting

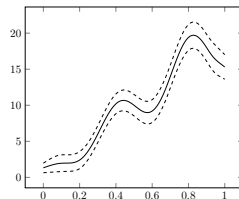
Procedure [Hua+98][Che+06][Hun]



(j) input signal s



(k) envelopes a_+, a_-



(l) r_1 mean of a_+, a_-

Calculate directly

$$u_0 = s - r_1$$

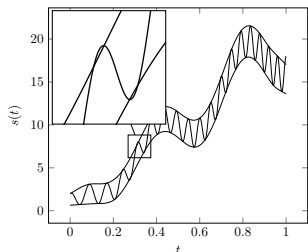
$$a_0 = a_+ - r_1$$

Phase ϕ_0 extracted from simplified IMF $u_0/a_0 = \cos(\phi_0)$.

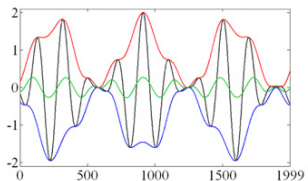
Envelope Estimation

Overview

- ▶ Classical method
(interpolation of local maxima) [Hua+98][Che+06]
(*intersection problem*) (m)
- ▶ Optimization based [HK13]
(*'cobweb effect'*) (n)
- ▶ Tangential point method
[HPH12]
(*tight conditions*)
- ▶ Local integral mean method
[Niu+21]
(*envelope estimation mixed with mean-derivation, engineering framework*)



(m) classical method



(n) optimization based, [HK13]

Iterative Slope Envelope Estimation [Hun]

Ansatz

WLOG: Upper envelope estimation of signal $s(t)$

Consider the classical method [Hua+98][Che+06]

1. Determine those \tilde{t} s satisfying $s'(\tilde{t}) = 0$ and $s''(\tilde{t}) < 0$ (local maxima)
2. Interpolate \tilde{t} s to obtain upper envelope estimate $m(t)$

Ansatz: Consider classical method only as one iteration of iterative process!

1. Set $m \equiv 0$ on the first run
2. Determine those \tilde{t} s satisfying $s'(\tilde{t}) = m'(\tilde{t})$ and $s''(\tilde{t}) < 0$
3. Interpolate \tilde{t} s to obtain new upper envelope estimate $m(t)$, goto 2

input : multicomponent signal $s \in \mathcal{C}^2([0, 1], \mathbb{R})$

tolerance $\varepsilon > 0$

output: upper envelope $m \in \mathcal{C}^2([0, 1], \mathbb{R})$

$m \leftarrow 0 \in \mathcal{C}^0([0, 1], \mathbb{R});$

repeat

$\tilde{m} \leftarrow m;$

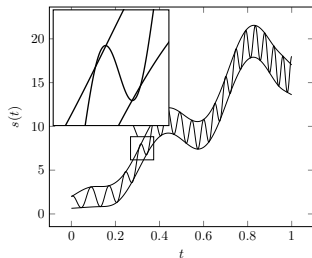
$P \leftarrow \{(t, s(t)) \in [0, 1] \times \mathbb{R} \mid (s - \tilde{m})'(t) = 0 \wedge (s - \tilde{m})''(t) < 0\};$

$m \leftarrow \text{Interpolate}(P \cup \{(0, s(0)), (1, s(1))\});$

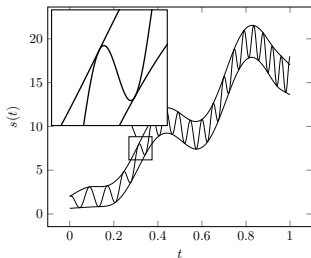
until $\| \max(s - m, 0) \|_\infty < \varepsilon;$

Iterative Slope Envelope Estimation [Hun]

Overview and Efficiency



(o) classical method



(p) iterative slope method

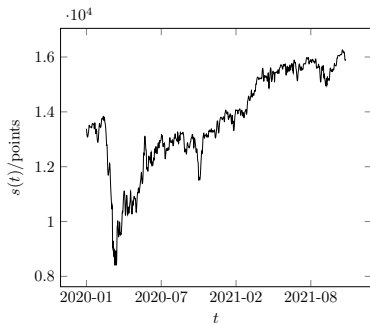
- ▶ Typically requires very few iterations to stabilize (\rightarrow example)
- ▶ Determination of interpolation points is embarrassingly parallel
- ▶ Natural cubic B-Spline-Interpolation is $\mathcal{O}(n)$ for n knots, knot count drastically decreases on each iteration

In total no asymptotic overhead compared to classical method

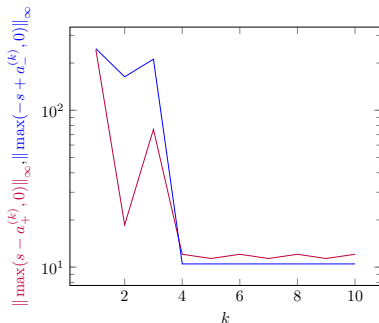
Iterative Slope Envelope Estimation [Hun]

DAX-Example (Overview)

DAX closing-prices from 2020-01-02 to 2021-11-25:



(q) signal $s(t)$

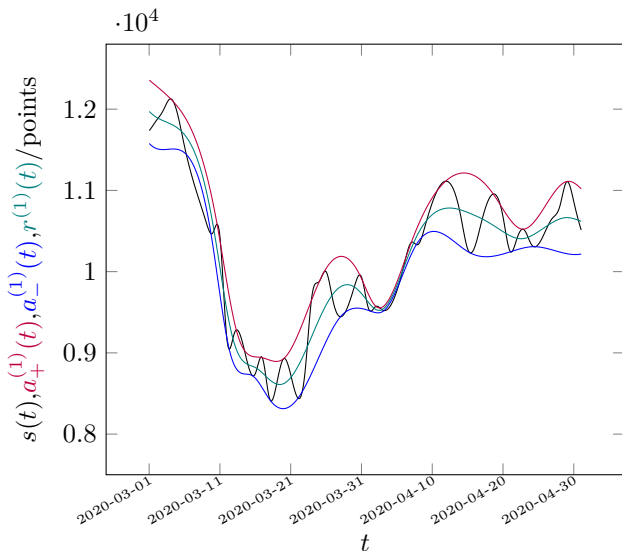


(r) convergence (for a_+ and a_-)

Iterative Slope Envelope Estimation [Hun]

DAX-Example (Classical Method)

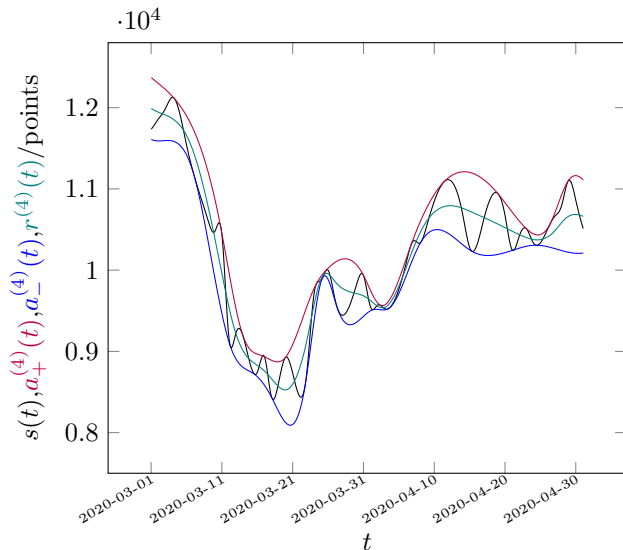
After first iteration (= classical method), **envelope intersection**:



Iterative Slope Envelope Estimation [Hun]

DAX-Example (Iterative Slope Method)

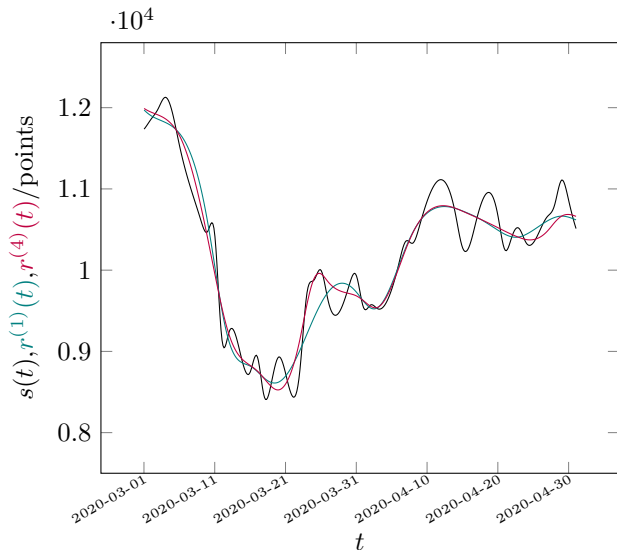
After 4 iterations, no envelope intersection:



Iterative Slope Envelope Estimation [Hun]

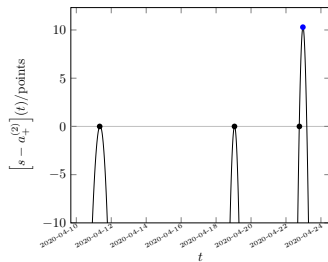
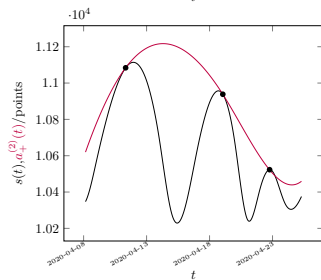
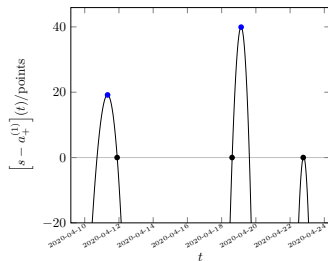
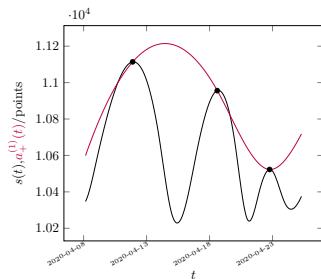
DAX-Example (Comparison of residuals)

Comparison of the ‘classical’ residual $r^{(1)}$ and ‘iterated’ residual $r^{(4)}$:



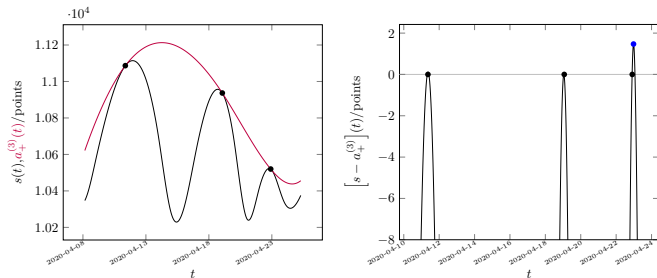
Iterative Slope Envelope Estimation [Hun]

DAX-Example (A Closer Look at Convergence (1/2))



Iterative Slope Envelope Estimation [Hun]

DAX-Example (A Closer Look at Convergence (2/2))



Observations

- ▶ Corrections become increasingly smaller
- ▶ Interpolation dampens effect of corrections on slope in other 'stabilized' areas
- ▶ Convergence is reasonable to assume in the general case
- ▶ Proof: Requires look at B-Spline-machinery to prove self-stabilization

Iterative Slope Envelope Estimation [Hun]

Overview and Outlook

Multivariate Generalization

- ▶ Follows canonically:

input : multicomponent signal $s \in \mathcal{C}^2([0, 1]^d, \mathbb{R})$
tolerance $\varepsilon > 0$

output: upper envelope $m \in \mathcal{C}^2([0, 1]^d, \mathbb{R})$

$m \leftarrow 0 \in \mathcal{C}^0([0, 1]^d, \mathbb{R});$

repeat

$\tilde{m} \leftarrow m;$
 $P \leftarrow \{(\mathbf{t}, s(\mathbf{t})) \in [0, 1]^d \times \mathbb{R} \mid \nabla(s - \tilde{m})(\mathbf{t}) = \mathbf{0} \wedge \mathbf{H}(s - \tilde{m})(\mathbf{t}) \prec 0\};$
 $m \leftarrow \text{Interpolate}(P \cup (\partial[0, 1]^d, s(\partial[0, 1]^d)));$

until $\|\max(s - m, 0)\|_\infty < \varepsilon;$

- ▶ Implementation for example via sparse tensor product B-Splines

EMD-Context

- ▶ Improvement on the envelope estimation heuristic, yields ‘true’ envelopes by construction
- ▶ Drop-in replacement for classical method